Differential Turns

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Turning competition between two aircraft represented in energy approximation is formulated as a differential game. The structure of the family of solutions is discussed and two limiting cases examined: that of very long duration, emphasizing steadily sustainable turn rates, and that of very short duration, dominated by instantaneously available rates.

Introduction

RECENT work on extensions of 3-D energy maneuverability theory has succeeded in fitting two earlier results, Preyss/Willes "corner velocity" optimum and the Boyd/Christie "corridor," previously thought contradictory, into a pattern.[1,2] However, the family of energy turns generated by this theory leaves a large upper portion of the Mach number-altitude envelope uncovered. Therefore, for air-combat maneuvering, these "one-on-zero" results are of limited use. An analogous modelling possibility for two craft opposed in combat might be called a *differential turning game.* Attention is focused on the difference in heading of pursuer and evader; the former wishes to minimize this difference some time later, and the latter to maximize it. Each "player" is presumed to have full knowledge not only of the other's state (specific energy and heading) but also of his vehicle's characteristics. Systematically exploiting the differences is the name of the game. Actually, two separate games are to be solved for families of turning maneuvers corresponding to various initial states; one with the two aircraft predesignated as pursuer and evader, the second with these roles reversed. These two families turn out to be identical, of course, in the special case of identical aircraft.

In a combat tail-chase, the initiative of altitude choice falls to the evader, and the pursuing aircraft must match this choice, more or less, in order to maintain visual contact. This assumption supplies the coupling between the two vehicles' motions that makes the important difference between the game and a mere turning contest, whose outcome would be apparent by comparison of energy-turn data for the separate craft generated with altitude programs individually optimized. This idealization of a dogfight as primarily a turning competition with the evader steering the chase up or down to his advantage seems fairly apt for certain phases of combat. An objection to the gaming format, that pilots do not fly anything like minimax optimally in combat, is true but irrelevant to the task of evaluating airplanes and weapons; a tactics-manual pilot model may be 80% of optimal in the Sabrejet but only 60% in the MiG. It is better, perhaps, to assume an exceptionally well-briefed pilot executing minimax-optimal tactics flawlessly. The turn tactics emerging should be of as much interest as the scoring, for they will tend to highlight strengths and weaknesses and their systematic exploitation.

Presented as Paper 72-950 at the AIAA Atmospheric Flight Mechanics Specialists Conference, Palo Alto, Calif., September 11–13, 1972; submitted September 18, 1972; revision received January 17, 1973. Sponsored by Headquarters USAF, Asst. Chief of Staff, Studies and Analysis, under contract F 44620-71-C-0123.

Index categories: Military Aircraft Missions; Aircraft Performance; Navigation, Control and Guidance Theory.

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Modeling

Attention focuses on the effort of the pursuer to align headings and of the evader to thwart it. Horizontal position coordinates are ignored, it being assumed that the circling motions as viewed from above are roughly concentric; this is thought descriptive of a large enough class of turning engagements to be of interest. The problem may be formulated as a differential game with three state variables: energies E_1, E_2, and $\Delta\chi = \chi_1 - \chi_2$, the difference in heading angles.

$$\dot{E}_1 = V_1(T_1\eta_1 - D_1)/W_1$$
$$\dot{E}_2 = V_2(T_2\eta_2 - D_2)/W_2$$
$$\Delta\dot{\chi} = gL_1 \sin\mu_1/V_1W_1 - gL_2 \sin\mu_2/V_2W_2$$

The analysis and computer program of Refs. 1 and 2 are adopted as a frame of reference. The Hamiltonian function is

$$H = \lambda_{E_1}\dot{E}_1 + \lambda_{E_2}\dot{E}_2 + \lambda_\chi \Delta\dot{\chi}$$

H is minimized by the evader's controls, the throttle, η_1, the bank angle, μ_1, and the altitude, h_1; and maximized by the pursuer's η_2 and μ_2. The pursuer's altitude is, in the first instance, taken as nearly equal to the evader's as constraints will allow. A lower limit on altitude is imposed as usual by one of the constraints $\beta_1 \geqq O$ (terrain), $\beta_2 \geqq O$ (dynamic pressure limit), $\beta_3 \geqq O$ (Mach no. limit), while an upper bound is imposed by a constraint $\beta_6 \geqq O$ representing a generalized "ceiling" (i.e., an altitude limit assuring nonnegative energy rate in level flight at the current value of specific energy).

$$\beta_6 = \bar{h}_6 - h \geqq O$$

where $\bar{h}_6 = \bar{h}_6(E)$ is a precalculated function. A basic problem version is that in which the pursuer is allowed zero leeway in altitude choice. Even in this case the mechanization of the search for the minimum of H vs h is not simple, as it has to contend with two breaks in slope (at the two corner-velocity altitude) and differing bounds on h for the two craft.

Bounds on the bank angles μ_1 and μ_2 corresponding to maximum lift coefficients and normal load factor limits apply and may be incorporated by use of the Valentine device as in Refs. 1 and 2.

Games of Long Duration

It is instructive to consider, as one limiting case, games of long, but prescribed, duration as these separate cleanly into three phases: an initial transient, a stabilized circling at constant but generally unequal turn rates, and a final transient. The steady-turn portion can be studied in terms of a minimax optimization for an ordinary game rather than a differential game, and can provide much intuitive insight. The solution can be found quite readily by a survey of the difference between steady-turn rates that can be developed by the two aircraft at various altitude Mach number combinations. The aircraft designated pursuer

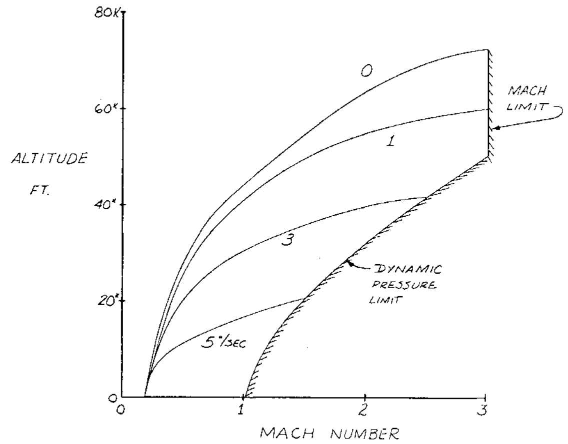

Fig. 1 Sustainable turn rates, aircraft A.

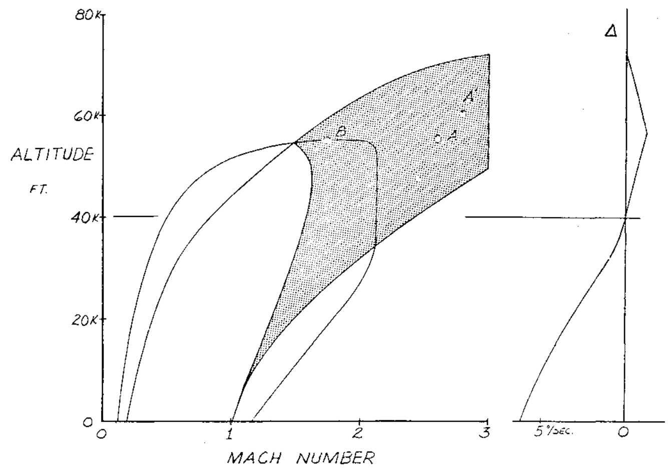

Fig. 3 Pursuer/evader, B/A.

chooses his own Mach number, while the evader chooses not only his Mach number but the altitude for both. One mins the turn-rate difference, the other maxes, and the steady-state game is solved by selection from the survey. By reversing the roles and freedoms, one may obtain a solution to the complementary turning game. An example is shown in the following section.

The initial transient is complex, involving a careful balance between turn rates and energy changes necessary to approach the steady-turn condition, carried out, in the case of the evader, with disadvantage to the pursuer as a main factor in the altitude choice. The computation volume generally will be more than twice that involved for single aircraft energy-turns[1,2] because of the interaction. The final transient represents conversion of energy just before the allotted time expires. In actual combat, this might be triggered on fuel depletion instead. For a typical combat encounter of 3–5 min, one may expect the transients generally to overlap, and the three-phase description to have little more than conceptual value.

The long-duration solutions just discussed tend to be characteristic of successful, or nearly successful, evasions, i.e., the chase does, in fact, reach that part of the state space favorable to the evader as far as sustainable turn rates are concerned. Short-term strategies and early capture represent the other extreme, typified by the situation of a pursuer having performance generally lower than the evader but starting out with a substantial energy advantage. In this circumstance, the evader clearly would prefer to satisfy an instantaneous state inequality on the angle difference over any long-term goal, should this be possible. Similarly, if the pursuer can indeed overtake, regardless of the evader's tactics, he would wish to maximize the time spent with this same inequality violated as it represents a measure of weapons-firing opportunity. A single problem statement including a state inequality can cover both short- and long-term cases in which the evader escapes, but an essentially different problem statement is needed to cover overtaking, i.e., the whole problem is really an Isaacs' "game of kind."[3] Which statement applies depends upon the initial state, and the separation boundaries are of interest. In the present investigation, which is exploratory rather than definitive, attention will be confined to the two simplest cases: capture with time minimized by the pursuer and maximized by the evader when capture occurs at all, and the heading difference minimaxed at some large, but fixed, final time when it does not.

Differential turn computations for well-matched aircraft will often extend through several revolutions. Since there is an obvious limit of 180° to the optimality of essentially horizontal turns for a single aircraft (e.g., in the constrained-horizontal case), some consideration is due the analogous differential-game phenomena, explored to some extent in Ref. 4, a suitable topic for future research. For the moment, we offer a crude analogy which may be suggestive. It is the pursuit/evasion problem on a sphere with constant-speed vehicles capable of instantaneous turns. With nearly equal speeds and vehicles initially far apart, there will evidently be a chase through multiple revolutions which is really minimax-optimal; while for the single vehicle problem, the optimality of a great circle arc breaks down at 180°. Of course, this is merely an illustration that "closed-loop" conjugate-point tests are less restrictive than open-loop tests, but it seems an apt one.

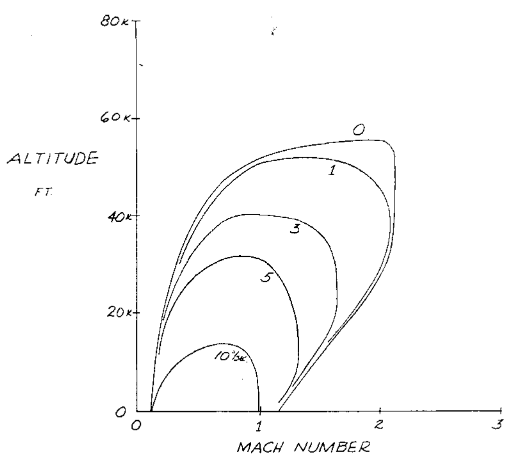

Fig. 2 Sustainable turn rates, aircraft B.

Steady Turn Minimax

Contours of sustainable turn rates in the Mach number-altitude chart are shown in Fig. 1 for Aircraft A, the hypothetical Mach 3 aircraft of Refs. 1 and 2. Similar data for Aircraft B, a version of the F-4 aircraft, appear in Fig. 2. The overlay of Fig. 3 shows the region of steady-turn rate superiority of Aircraft A shaded. The difference of best turn rates at each altitude is plotted to the right, i.e., the pursuer, B, has minimized this difference by choice of his Mach number, while the evader, A, has maximized it by choice of his. The remaining choice, that of altitude, falls to the evader, A, who takes it so as to maximize the difference, plotted on the right. The Mach number/altitude combinations corresponding to this minimax point are the circles labelled A and B. The results for a similar minimax carried out with the evader given 5000-ft-alt leeway are labelled A′ and B.

The overlay of Fig. 4 has the region of Aircraft B's steady turn superiority shaded. A minimax computation with A in the pursuit role produces the two circles at 0 alt, as labelled.

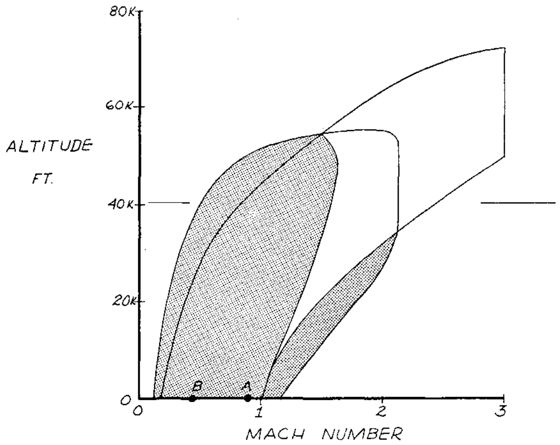

Fig. 4 Pursuer/evader, A/B.

While the main interest for present purposes is to illustrate the minimax viewpoint and provide background for the differential turning game, it is thought that the minimax operating points plus the horizontal line marking the shift altitude might be useful adjuncts to standard overlay charts.

Capture Closing Rate Minimax

Another limiting case of interest is that of very short duration. Consider the sign of the turn-rate difference at the instant of "capture," which we define to mean zeroing of the state inequality on heading with subsequent violation at the pursuer's option. Clearly $\Delta\dot{\chi}$ for overtaking must be negative when minimized by the pursuer, even when maximized by the evader. In the chart whose axes are the two specific energies, regions of negative minimaxed $\Delta\dot{\chi}$ are possible capture regions. The boundaries of these regions are of special interest in the computation of families of solutions, as they provide a key to determining the surface in E_1, E_2, $\Delta\chi$ space that separates successful pursuits from successful evasions.

Figure 5a shows the situation for identical aircraft (Aircraft B) with an assumed rigid requirement for the pursuer to match altitudes. The minimaxed turn-rate difference is positive over most of the energy diagram—zero along the 45° line of equal energies, and negative only in the shaded region that corresponds to an energy advantage by the pursuer when the evader's energy level is below that for the sea-level corner velocity. Thus, low and slow spells terminal-phase vulnerability.

The turn-rate difference depends upon wing areas, limit load factors, and maximum lift coefficients as functions of Mach number, but not at all on thrust or drag, as energy rates are unrestricted in this computation. Indeed, a well-stressed sailplane design would look superior to a high-performance opponent in this type of chart. The evader's corner-velocity locus in the Mach number-altitude chart and the associated turn rates govern at energies down to that of the evader's corner velocity at sea level. Above this value, the evader chooses altitude so as to turn at his own corner velocity. At energies below the sea-level corner value, the minimum altitude restriction foils any such tactic, the action is entirely at zero altitude, and an energy advantage, but not too great a one, is needed by the pursuer for closure.

Figure 5b shows the result of removing the altitude-match requirement. Figure 5c is for an intermediate case with 1000 ft altitude leeway allowed. The pursuer will generally employ the altitude freedom to operate closer to his own corner-velocity altitude. These tactics are, of course, not even qualitatively applicable at times well before capture, since the minimax of turn-rate difference is done in the present computations with complete disregard of energy rates; rather, a compromise between

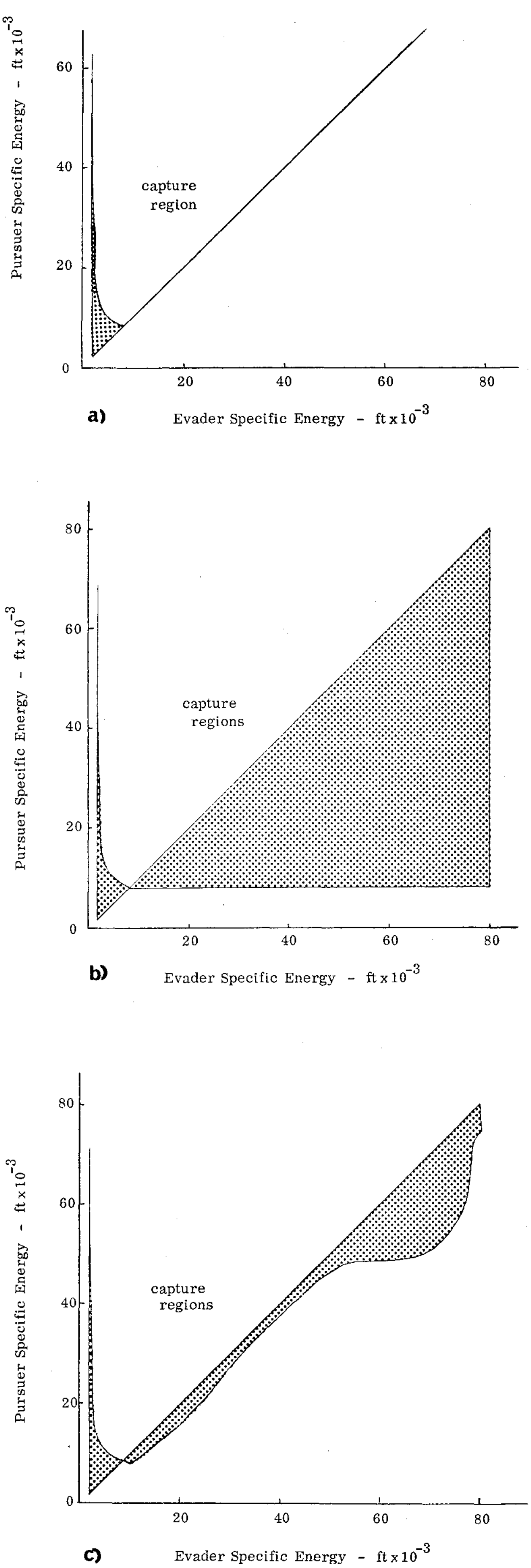

Fig. 5 Energy regions at capture, pursuer/ evader, B/B; a) equal altitudes, b) unrestricted altitude, and c) altitude leeway 1000 ft.

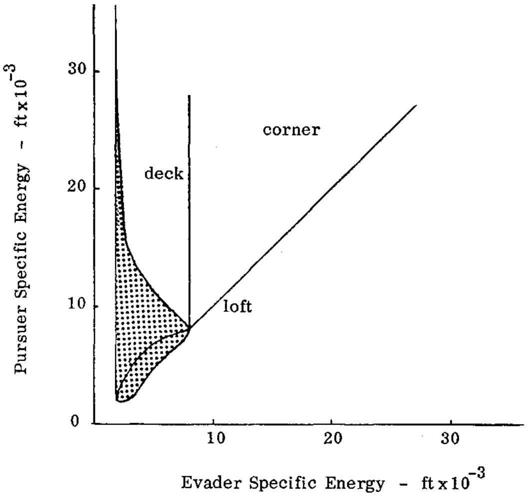

Fig. 6 Energy regions at capture, pursuer/evader, A/B: equal altitudes.

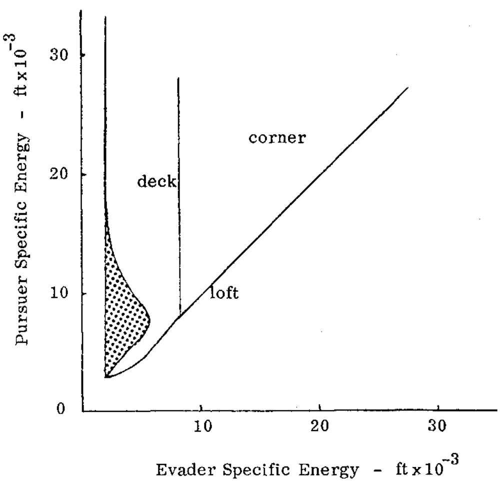

Fig. 7 Energy regions at capture, pursuer/evader, B/A: equal altitudes.

turn and energy rates analogous to that of the Boyd/Christie "corridor" for a single aircraft[5] can be expected.

Figure 6 describes terminal closure with matched altitudes for nonidentical aircraft; A is the pursuer here and B the evader. The capture region is again shaded and it does not extend to the right of the evader's sea-level corner-velocity energy. It will generally be the case, with a matched-altitude requirement, that the capture region does not extend to high evader energies unless the pursuer has superior lifting capability not only in terms of structural g-limit but also aerodynamically, as measured by the quotient of wing loading and maximum lift coefficient. Figure 7, for B chasing A, altitude matched, is seen to be qualitatively similar. Both Figs. 6 and 7 indicate the evader's tactics in respect to altitude choice in various regions. The label deck denotes zero altitude, corner denotes choice of the evader's corner-velocity altitude, and loft indicates operation at high altitude so as to run the pursuer out of lift or power.

An energy advantage of pursuer over evader is necessary for capture when the turning chase has spiralled down to low altitude and subsonic speeds, as it typically does, but it tends to be a liability in early attempts at closure as long as the evader exploits the freedom of altitude choice.

Computational Approach

Numerical solution of the state/Euler system in the spirit of the classical indirect method is contemplated, generally as carried out for single aircraft turns in Ref. 1. Since the specific energies are unspecified at the final point, the corresponding multiplier variables vanish there by transversality; the final value of the function H is -1, also by transversality. Since the family of successful pursuits with initial E_1, E_2, and $\Delta\chi$ varying over wide ranges is of interest, as one example, computation of members by backward integration proceeding from points within the shaded capture set seems attractive. The boundary of the region (characterized by $\Delta\dot{\chi} = 0$) is exceptional in that no amount of scaling of λ_χ can produce $H \neq 0$ and Euler solutions emanating from such points are abnormal, in classical variational terminology. Such points correspond to capture with final time unspecified, i.e., capture barely attainable at all. Extremals integrated backwards in time from such points (which will then exhibit $H = 0$ for all t, since H is constant) will be of particular value in defining the boundary of the region in E_1, E_2, $\Delta\chi$ space that separates successful pursuits from successful evasions. The proximity of the altitude switching surface, suggested by Figs. 6 and 7, could be troublesome computationally. The appearance of singular surfaces of types peculiar to differential games[6] is difficult to anticipate, but would represent a major complication. Unfortunately, a criterion analogous to that of Ref. 7 for ruling out the appearance of such surfaces a priori is not available for so complex a model as the present one.

Controls from complete turn-maneuver families in feedback form represent pursuit and evasion tactics. These might be approximated as stored functions of the two specific energies and the heading difference for simulation and training display purposes. The evader's altitude must be treated as a command value for manual or automatic flight control execution.‡

Conclusions

Computation of differential turn families along the lines sketched here is of future interest. Simulations carried out with tactics so obtained and approximated in feedback form would be appropriate to determine the limits of the modelling approximations, i.e., the order-reduction and the neglect of horizontal position effects. Another application for complete families of both games would be construction of dogfights of a sort by piecing together segments with role reversals of pursuer and evader at 180° opposition.

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‡ There is a performance index for this, that falls out of the asymptotics.[2]